ISRAEL JOURNAL OF MATHEMATICS, Vol. 39, No. 3, 1981

REMARKS ON A PRINCIPLE OF LOCALIZATION

BY

SHMUEL ROSSET

ABSTRACT

We prove that the Brauer class of a crossed product is a sum of symbols iff its "local" components are. Analogously we show that a solution of the "Goldie rank conjecture" would follow from the "local" statements; an extension of a result of Cliff–Sehgal is an easy corollary.

1. Let $L \supset K$ be a finite galois extension of fields, $G = \mathrm{Gal}(L/K)$, and let $f: G \times G \to L^*$ be a (non-homogeneous) 2-cocycle. Let $A = (L/K, f)$ be the crossed product defined by this situation [4]. It is a simple algebra with center K and dimension n^2 ($n = |G|$) over K. As such it is isomorphic to an algebra of the type $M_r(D)$, where D is a division algebra over (i.e. with center) K. We call r the rank (or Goldie rank, denoted rk) and if $(D:K) = d^2$ then d is the index. Note that the index of an algebra is an invariant of its Brauer class. The number (rank) $\cdot$ (index) is called the degree.

For each prime number p dividing n let G_p be a (fixed) p-sylow subgroup of G and $K_p = L^{G_p}$, the fixed field under the action of G_p. Let f_p be the restriction of f to $G_p \times G_p$ and let $A_p = (L/K_p; f_p)$. We call A_p the (p-) local component of A. If the primary decomposition of $[A]$ in $\mathrm{Br}\, K$ is $\Sigma_p [D_p]$, where D_p is a division algebra over K with $\deg(D_p)$ a power of p, we call D_p (or $[D_p]$) the (p-) primary component of A. (Note that the group operation in $\mathrm{Br}(\cdot)$ is denoted by $+$.)

It is obvious that $[A_p] = \mathrm{res}_{K_p/K}[A]$. Since K_p splits D_q for $q \neq p$ we see that

$$[A_p] = \mathrm{res}_{K_p/K}[D_p].$$

Since $\mathrm{cor} \cdot \mathrm{res}_{K_p/K} =$ multiplication by $(K_p : K)$ we see that $[D_p]$ is a multiple of $\mathrm{cor}_{K_p/K}[A_p]$, by an integer prime to p. In particular they have the same index and since $[A_p] = [D_p \otimes_K K_p]$ and $(p, (K_p : K)) = 1$ this is also the index of $[A_p]$. Thus we can prove the following simple observation:

Received June 30, 1980

LEMMA 1. (i) $d = \Pi_{p/n}\,\mathrm{index}(A_p)$. (ii) $r = \Pi_p\,\mathrm{rk}(A_p)$.

PROOF. (i) is proved above and (ii) is a trival consequence.

We can use [7] to show

LEMMA 2. *If K contains a primitive n-th root of 1 (so $\mathrm{char}\,K \neq n$) and each $[A_p] \in \mathrm{Br}(K_p)$ is a sum of cyclic algebra classes* ("symbols" in the terminology of [7]), *then $[A]$ is a sum of cyclic algebras.*

PROOF. In [7] it is shown that under these assumptions the corestriction preserves sums of cyclic algebras.

COROLLARY. *If G is a group all of whose Sylow subgroups are cyclic, then any crossed product with group G (over a center containing a primitive $|G|$-th root of 1) is a sum of cyclic algebras.*

This corollary was proved, quite differently, by Snider [8].

2. We now give a different proof of Lemma 1, a proof which generalizes to group rings of virtually polycyclic groups.

Let χ be a rational valued function on the category of finitely generated A modules which is additive, i.e., $\chi(M) = \chi(M') + \chi(M'')$ whenever $0 \to M' \to M \to M'' \to 0$ is exact ("Euler characteristic") and normalized by $\chi(A) = 1$. If N is a simple non-zero A module then every finitely generated A module is of the type N^s and it follows from $A \cong N^r$ $(A = M_r(D))$ that χ is unique and is defined by $\chi(N^s) = s/r$. We can formulate this:

LEMMA 3. *The rank of a central simple algebra is the least common multiple of the denominators of the numbers $\chi(M)$ written in reduced form.*

We must show that the p-part of r is the number r_p such that $A_p \cong M_{r_p}(D(p))$. The dimension of A over $D(p)$ is easily seen to be $r_p^2 \cdot n'_p$ where $n'_p = (G : G_p)$. The function on A modules defined by $M \mapsto \dim_{D(p)}(M)/(r_p^2 \cdot n'_p)$ is additive and is 1 on A. Hence it is equal to χ. Since the simple A_p module has dimension r_p over $D(p)$ we see that, in reduced form, the p part of the denominator of $\dim_{D(p)}(M)/(r_p^2 \cdot n'_p)$ is, at most, r_p. It remains to show that the p-part of r is precisely r_p. For this it is enough to exhibit an A module M with $\chi(M) = 1/r_p$. Let χ_p be the analogue of χ for A_p, i.e., χ_p is the unique function on A_p modules which is additive and is 1 on A_p. If V is an A_p module the function φ defined by $\varphi(V) = \chi(A \otimes_{A_p} V)$ is such a function, so $\varphi = \chi_p$. It follows that if V is a simple (non-zero) A_p-module then $1/r_p = \chi_p(V) = \chi(A \otimes_{A_p} V)$, so we have exhibited the M with $\chi(M) = 1/r_p$.

REMARK. Somewhat similar dimension computations appear in: Jacobson, *Structure of Rings*, Ch. VI.

3. In this section we deal with *virtually polycyclic* groups (also called polycyclic-by-finite). These are groups which have a subgroup of finite index which is polycyclic. A general reference is Passman's book [5]. We will denote these groups by "*VP-groups*".

A VP-group contains a normal poly-{infinite cyclic} (or poly-**Z**, for short) subgroup of finite index [5, p. 422]. The group ring, over a field k, of a poly-**Z** group has finite global homological dimension [5, p. 626] and every finitely generated projective module is stably free. In particular, and using the noetherianity of such group rings, every finitely generated module has a finite free resolution.

Let Γ be a VP group, Γ' a poly-**Z** subgroup of finite index, and let k be a field. If M is a finitely generated $k\Gamma$ module let M' be M considered as a $k\Gamma'$ module only. Let $0 \to F_m \to \cdots \to F_1 \to F_0 \to M' \to 0$ be a resolution of M' over $k\Gamma'$, where F_j is free over $k\Gamma'$ of rank f_j. We define $\chi_\Gamma(M) = (\Gamma : \Gamma')^{-1} \cdot \Sigma(-1)^j f_j$. This is independent of the resolution and of the choice of Γ'. It is an Euler characteristic, i.e., additive on exact sequences, and takes the value 1 on the free rank 1 module $k\Gamma$. We define the *rank* (or "*Goldie rank*") of Γ to be the least common multiple of the denominators of the numbers $\chi(M)$ written in reduced form (M being an arbitrary finitely generated $k\Gamma$ module).

Assume now that Γ' is normal in Γ and let $\Gamma/\Gamma' = G$. For each prime number p let G_p be a p-sylow subgroup of G and Γ_p its inverse image in Γ. Let $\chi_p(M)$ be the Euler characteristic of M thought of as a Γ_p module. Obviously

$$\chi_p(M) = (\Gamma : \Gamma_p)\chi(M)$$

which proves that the p-part of $\chi_p(M)$ is the same as the p-part of $\chi(M)$, hence independent of the choice of Γ'. We call $\chi_p(M)$ the $(p-)$ local component of $\chi(M)$ and the least common denominator of the numbers $\chi_p(M)$ (M an arbitrary f.g. $k\Gamma$ module) is called the $(p-)$ local part of the (Goldie)-rank. Clearly the rank divides the product of all local ranks.

Morover, we claim the Goldie rank is multiplicative, i.e.

LEMMA 4. $r_k(\Gamma) = \Pi_p r_k(\Gamma_p)$.

To prove this we use the following fact, proved in [6]. Let H be a subgroup of Γ, let N be a finitely generated kH module, and let $M = k\Gamma \otimes_{kH} N$ be the induced $k\Gamma$ module. Then $\chi_H(N) = \chi_\Gamma(M)$. In other words, χ is compatible with induction. It follows that the set of denominators in the (reduced) Euler characteristics of $k\Gamma_p$ modules is also a set of denominators for $k\Gamma$ modules. Hence $r_k(\Gamma_p) \mid r_k(\Gamma)$ and it is easily shown now that $r_k(\Gamma_p)$ is just the p-part of $r_k(\Gamma)$.

Using the (non-trivial) result that $r_k(\Gamma) = 1$ iff Γ is torsion free we obtain:

COROLLARY. *If $p \mid r_k(\Gamma)$ then Γ has p-torsion (p is a prime).*

Indeed by Lemma 4 p must divide $r_k(\Gamma_p)$ so Γ_p cannot be torsion free. But Γ_p can only have p-torsion.

Another proof of the Corollary, when $\mathrm{char}(k) = 0$, appears in [2].

Finally it should be mentioned that the Goldie rank defined here and the more usual Goldie rank coincide when $k\Gamma$ is prime, i.e., when Γ has no non-trivial finite normal subgroups [5, ch. 10]. Again the reader is referred to [6] for details and proof.

REFERENCES

1. G. H. Cliff, *Zero divisors and idempotents*, preprint.
2. G. H. Cliff and S. K. Sehgal, *On the trace of an idempotent in a group ring*, Proc. Amer. Math. Soc. **62** (1977), 11–14.
3. D. R. Farkas and R. L. Snider, *K_0 and Noetherian group rings*, J. Algebra **42** (1976), 192–198.
4. I. N. Herstein, *Noncommutative Rings*, Carus Math. Mon. #15.
5. D. S. Passman, *The Algebraic Structure of Group Rings*, Wiley-Interscience, New York, 1977.
6. S. Rosset, *The Goldie rank of virtually polycyclic groups*, to appear.
7. S. Rosset, *The corestriction preserves sums of symbols*, to appear.
8. R. L. Snider, *Is the Brauer group generated by cyclic algebras?*, in *Ring Theory, Waterloo 1978*, LN 734, Springer Verlag.

DEPARTMENT OF MATHEMATICAL SCIENCES
TEL AVIV UNIVERSITY
RAMAT AVIV, ISRAEL